JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 17, No. 1, January–February 1994

Power Limited Soft Landing on an Asteroid

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This work is concerned with the final approach phase and vertical landing on a spherical asteroid with a power limited, electrically propelled spacecraft. With gravitational effects taken into account, a new solution to the fuel optimal vertical landing on an asteroid was obtained. In this particular solution the spacecraft commanded acceleration is colinear with the vehicle velocity, and the spacecraft trajectories are restricted to a plane. Based on qualitative methods of analysis, the guidance strategy and the resulting trajectories were studied. It is shown that these fuel optimal trajectories effectively assure a vertical soft landing on the asteroid. Results of numerical simulations for the vertical landing starting from an elliptic orbit are presented.

I. Introduction

FOR the past 30 years mission feasibility studies and trajectory analyses have been conducted to assess the possibility of a rendezvous with small celestial objects such as comets and asteroids. The interest in this kind of mission stemmed both from its scientific importance and its possible practical applications. Asteroids form an ordered assemblage of protoplanetary fragments that seem to remain near the original locations of their formation between Mars and Jupiter. The compositional and structural variations of objects in the asteroid belt are thought to retain clues about the primordial conditions and processes that contributed to the formation of the solar system. An ideal mode of an asteroid mission for scientific purposes would be to achieve rendezvous with a diverse set of asteroids during a single mission. Trajectory options available for a modified Mariner spacecraft mission to asteroids were presented in Ref. 1. Missions for a comet-nucleus sample return mission using electric propulsion at very low thrust levels were studied in Ref. 2. In Ref. 3 the placement of transmitters on selected asteroids to provide an independent navigation capability for interplanetary flight was proposed. This interest in a rendezvous with small celestial objects provided the motivation for the present work.

Rendezvous missions to celestial objects are in general divided into a number of navigation and guidance phases. Works such as Refs. 4–6 were concerned with the optimization of transfer orbits with power limited, electrically propelled spacecraft. Works on the final approach and landing on a celestial object without atmosphere[7] obtained fuel optimal trajectories under the assumption of a constant acceleration of gravity.

For landing on a small celestial object such as an asteroid, even though the acceleration in absolute terms is small, its variation is very large and should be explicitly taken into account.

In this work the final approach phase and landing on an asteroid of a power limited electrically propelled spacecraft, with gravitational effects taken into account, are studied. A particular solution to the fuel optimal vertical landing is obtained.

Received June 6, 1991; revision received Aug. 10, 1992; accepted for publication Nov. 14, 1992.

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II. Problem Definition

Rendezvous and landing on a small celestial object without atmosphere are particular cases of the more general problem of transfer. The optimal transfer problem can be defined in its most general form as follows: Given a vehicle in a central gravitational field at initial position $\boldsymbol{r}_o$ and velocity $\boldsymbol{v}_o$, transfer this vehicle to a final position $\boldsymbol{r}_f$ and velocity $\boldsymbol{v}_f$ at time t_f and keep fuel expenditure to a minimum. The final t_f can be given or free.

The system equations in vectorial form and Cartesian coordinates are given by

$$\dot{\boldsymbol{r}} = \boldsymbol{v} \tag{1}$$

$$\dot{\boldsymbol{v}} = \boldsymbol{a} - \frac{\mu \boldsymbol{r}}{r^3} \tag{2}$$

where $\boldsymbol{r}$, $\boldsymbol{v}$, and $\boldsymbol{a}$ are, respectively, the position, velocity, and applied acceleration vectors of the vehicle under control and μ is the gravitational constant of the central body. The specific problem considered in this work is to find the required acceleration for the transfer of a spacecraft from an elliptic orbit about the celestial body to a soft landing on the body surface (zero relative velocity), and minimize fuel expenditure. The spacecraft employs an electric engine to generate the required accelerations.

III. Optimality Criterion

Electric engines generate thrust by accelerating charged particles through an electromagnetic field. In an electric engine it is possible to modify both the ejection speed v_c and the mass flow rate $\dot{m}$, and therefore to control independently the thrust T and power P, given by

$$T = -\dot{m} v_c \tag{3}$$

$$P = \tfrac{1}{2} |\dot{m}| v_c^2 \tag{4}$$

From these relations follows

$$\frac{1}{m_f} - \frac{1}{m_o} = \int_{t_o}^{t_f} \frac{a^2}{2P}\, \mathrm{d}t \tag{5}$$

where m is mass and $a = T/m$ is the applied thrust acceleration. For a vehicle with electric propulsion, we have a limited power (LP) system. The optimal operating point of an LP system uses the maximum power, and the optimization criterion to minimize the fuel is to minimize the quadratic performance index

$$J = \frac{1}{2} \int_{t_o}^{t_f} a^2\, \mathrm{d}t \tag{6}$$

The problem considered here is to transfer a vehicle from state (r_o, v_o) to state (r_f, v_f) at time t_f such that the performance index J is a minimum with t_f free.

IV. Optimal Control

The maximum principle of Pontryagin is employed.[8] The Hamiltonian of the system is given by

$$H = p_r \cdot v + p_v \cdot (a - \mu r/r^3) + p_J a^2/2 \tag{7}$$

where vectors p_r, p_v, and scalar p_J are the components of the adjoint vector and are defined by

$$\dot{p}_r = \frac{\mu}{r^3}\left[p_v - 3(p_v \cdot r)\frac{r}{r^2}\right] \qquad p_r(t_f) = p_{rf} \tag{8}$$

$$\dot{p}_v = -p_r \qquad p_v(t_f) = p_{vf} \tag{9}$$

$$\dot{p}_J = 0 \qquad p_J(t_f) = -1 \tag{10}$$

From Eq. (10) follows immediately

$$p_J \equiv -1 \tag{11}$$

An extremal for the spacecraft acceleration is obtained by minimizing the Hamiltonian with respect to a, i.e.,

$$\frac{\partial H}{\partial a} = -a + p_v = 0 \tag{12}$$

It follows then that

$$a = p_v \tag{13}$$

where p_v is the so called primer vector.[9]

We shall now solve for p_v as a function of the state variables, their final values, and time.

For the free time case

$$H^* = 0 \tag{14}$$

Substituting Eqs. (2) and (9) into Eq. (7), it follows that

$$H^* = -\tfrac{1}{2} p_v^2 + \dot{v} \cdot p_v - \dot{p}_v \cdot v = 0 \tag{15}$$

Let us assume a solution for the vector p_v of the form

$$p_v = f(t)v \tag{16}$$

where $f(t)$ is a scalar function of time, to be defined. Substituting Eq. (16) into Eq. (15) and rearranging

$$-v^2\left[\tfrac{1}{2} f^2(t) + \dot{f}(t)\right] = 0 \tag{17}$$

From where it follows

$$\tfrac{1}{2} f^2(t) + \dot{f}(t) = 0 \tag{18}$$

Integrating with c as a constant of integration

$$f(t) = \frac{2}{c+t} \tag{19}$$

Substituting Eq. (19) into Eq. (16)

$$p_v = \frac{2}{c+t} v \tag{20}$$

In Appendix A it is shown that this particular solution for p_v effectively verifies the adjoint Eqs. (8) and (9).

The adjoint vector p_v, as defined by Eq. (20), provides a particular solution for the extremal acceleration vector as a function of the vehicle velocity and time

$$a^* = \frac{2}{c+t} v \tag{21}$$

This particular solution has the following two main characteristics:

1) Since the extremal acceleration vector is colinear with the vehicle velocity, the entire trajectory will rest in the plane defined by the initial vectors r_o and v_o. This extremal acceleration vector provides a solution only to two-dimensional problems.

2) This solution includes a single free parameter c, besides the final time, to match the required final conditions in position and velocity.

We shall now restrict our analysis to the two-dimensional case.

V. Two-Dimensional Case

The system equations in polar coordinates for the two-dimensional case with the acceleration as given in Eq. (21) are

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{2r\dot{\theta}}{c+t} \tag{22}$$

$$\ddot{r} - r\dot{\theta}^2 = \frac{2\dot{r}}{c+t} - \frac{\mu}{r^2} \tag{23}$$

where r and θ define the spacecraft position with respect to the central body.

Multiplying Eq. (22) by r and rearranging

$$\frac{\mathrm{d}r^2\dot{\theta}}{\mathrm{d}t} = \frac{2r^2\dot{\theta}}{c+t} \tag{24}$$

The angular momentum is defined by

$$h = r^2\dot{\theta} \tag{25}$$

Substituting Eq. (25) into Eq. (24), it follows that

$$\dot{h} = \frac{2h}{c+t} \tag{26}$$

Integrating from initial conditions $t = 0$, $h = h_o$, it is obtained

$$h = h_o \frac{(c+t)^2}{c^2} \tag{27}$$

From Eq. (27) it follows that to transfer from an initial orbit with angular momentum h_o to a final orbit with angular momentum h_f at $t = t_f$, c is given by

$$c = \frac{t_f}{\pm(h_f/h_o)^{1/2} - 1} \tag{28}$$

For $h_f/h_o < 1$, $c < 0$, and consequently from Eq. (21) the commanded acceleration is aligned but opposite to the vehicle velocity. In particular, for the case of landing on the central body, $h_f = 0$ and $c = -t_f$. This specific case will be further developed in the sequel.

Substituting $\dot{\theta}$ from Eqs. (25) and (27) into Eq. (23) it follows that

$$\ddot{r} - \frac{2\dot{r}}{c+t} + \frac{\mu}{r^2} - \frac{h_o^2(c+t)^4}{r^3c^4} = 0 \tag{29}$$

This nonlinear differential equation with time dependent coefficients for the range r has no known analytic solution. To analyze the problem further, qualitative methods of analysis will be employed.

VI. Qualitative Analysis

We shall restrict our analysis fo the particular case

$$c = -t_f \tag{30}$$

For $h_o = 0$, Eq. (29) has the particular solution

$$r = (9\mu/14)^{1/3}(t_f - t)^{2/3} \tag{31}$$

as can be directly verified by substitution.

Based on this particular solution, let us define two new variables p and σ instead of r and t, as follows:

$$p = r(t_f - t)^{-2/3} \tag{32}$$

$$e^{-\sigma} = (t_f - t)/t_f \tag{33}$$

With these new variables the following equation is obtained:

$$\frac{d^2p}{d\sigma^2} + \frac{5}{3}\frac{dp}{d\sigma} - \frac{14}{9}p + \frac{\mu}{p^2} - h_o^2 t_f^{-5/3}\frac{e^{-7\sigma/3}}{p^3} = 0 \tag{34}$$

Equation (34) is still nonlinear, however for $h_o = 0$ it becomes autonomous, and well known methods of analysis can be employed.[10] The qualitative analysis of Eq. (34) will in consequence be performed in two steps. First the particular case $h_o = 0$, corresponding to the rectilinear single dimensional case is considered.

The system defined by Eq. (34) for $h_o = 0$ has a single equilibrium point at

$$p = (9\mu/14)^{1/3} \qquad dp/d\sigma = 0 \tag{35}$$

This equilibrium point corresponds in terms of the original variables r and t to the particular solution given in Eq. (21). This equilibrium point is a saddle point, and the resulting trajectories and separatrices in the phase plane p, $dp/d\sigma$ are shown in Fig. 1.

Two classes of trajectories for p, $dp/d\sigma$ can be distinguished, depending on whether the initial conditions p_o, $dp/d\sigma|_o$ belong to one side or the other of separatrix S_1.

For $\sigma \to \infty$, the trajectories' behavior is as follows

1) $\lim_{\sigma\to\infty} p = \infty \qquad \lim_{\sigma\to\infty} dp/d\sigma = \infty$

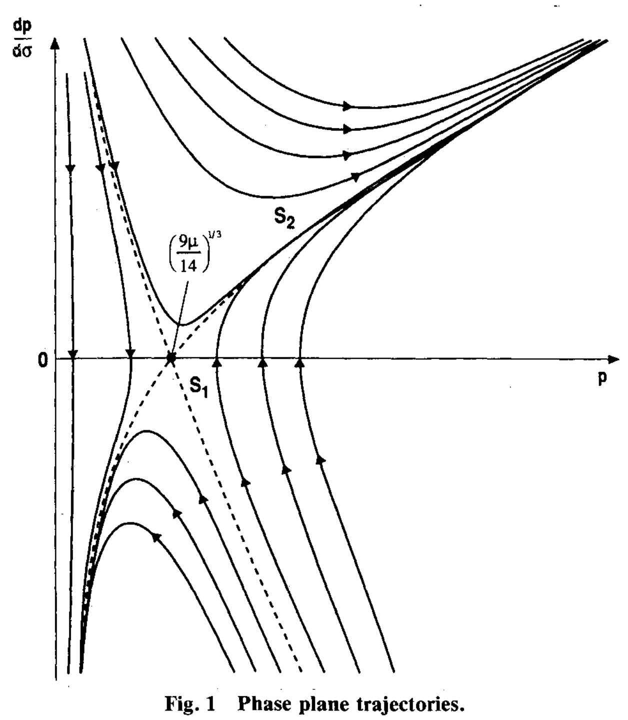

Fig. 1 Phase plane trajectories.

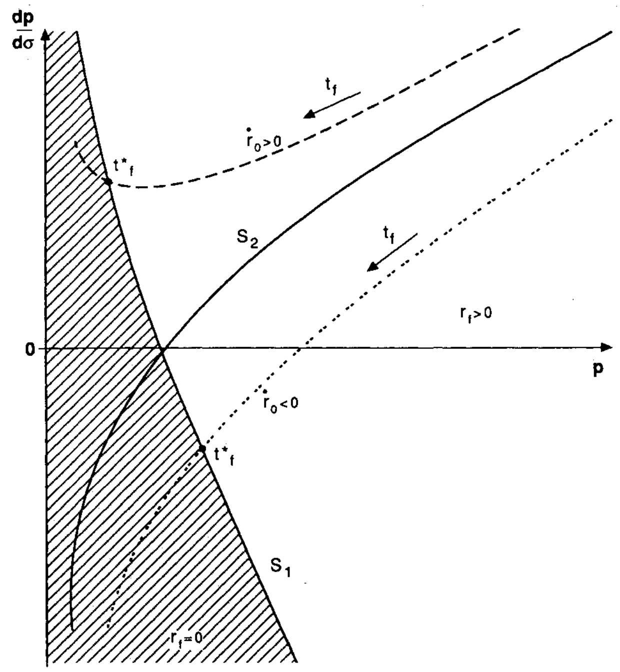

Fig. 2 Phase plane division.

2) $\lim_{\sigma\to\infty} p = 0 \qquad \lim_{\sigma\to\infty} dp/d\sigma \to -\infty$

Furthermore, for the trajectories of class 1, the exponential order of p is $2/3$, i.e., for $\sigma \to \infty$, p behaves as $e^{2/3\sigma}$.

To determine the behavior of the original variables, let us express the radial distance r and the radial relative velocity $\dot{r}$ in terms of p and σ

$$r = t_f^{2/3} p e^{-2\sigma/3} \tag{36}$$

$$\dot{r} = t_f^{-1/3}\left(\frac{dp}{d\sigma} - \frac{2}{3}p\right)e^{\sigma/3} \tag{37}$$

According to the behavior of p for $\sigma \to \infty$, it follows by direct substitution that for the trajectories of class 1

$$\lim_{t\to t_f} r = r_f$$

$$\lim_{t\to t_f} \dot{r} = 0$$

For trajectories of class 2

$$\lim_{t\to t_f} r = 0$$

$$\lim_{t\to t_f} \dot{r} = -\infty$$

Now, given initial conditions for the radial range and radial range rate at $t = 0$, r_o and $\dot{r}_o$, respectively, it follows from Eqs. (32), (33), (36), and (37) that

$$p_o = t^{-2/3} r_o \tag{38}$$

$$\left.\frac{dp}{d\sigma}\right|_o = t_f^{-2/3}(t_f\dot{r}_o + 2/3\, r_o) \tag{39}$$

It follows then, that for given values of r_o and $\dot{r}_o$, the initial conditions p_o and $dp/d\sigma|_o$ are functions of t_f.

In Fig. 2 the curves p_o, $dp/d\sigma|_o$ with t_f as a parameter are depicted for given values of r_o and $\dot{r}_o$. Depending on the value of t_f, the initial conditions in the plane p, $dp/d\sigma$ are located either on one side or the other of separatrix S_1, giving rise to trajectories of class 1 or 2, accordingly. The intersection of the

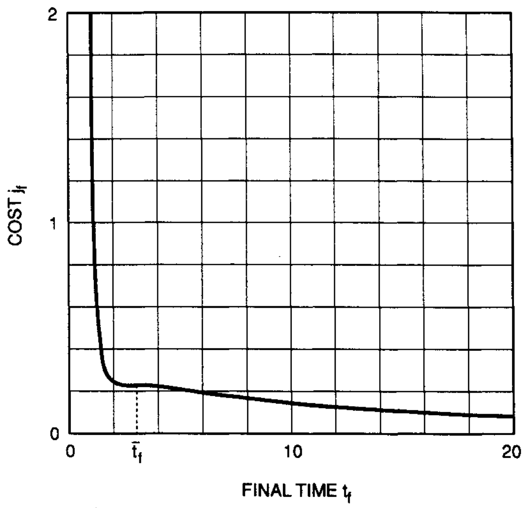

Fig. 3 No gravity, rendezvous cost J_f vs final time t_f.

parametric curve $p_o(t_f)$, $dp/d\sigma|_o(t_f)$ with the separatrix S_1, defines a value of $t_f = t_f^*$, such that

1) For $t_f < t_f^*$ $\lim_{t \to t_f} r = r_f$ $\lim_{t \to t_f} \dot{r} = 0$

2) For $t_f > t_f^*$ $\lim_{t \to t_f} r = 0$ $\lim_{t \to t_f} \dot{r} = \infty$

The optimal guidance law achieves a soft rendezvous at $r = r_f$, provided $t_f < t_f^*$.

Once the particular case $h_o = 0$ has been analyzed, the general case as defined by Eq. (34) is now considered. Since the nonlinear time varying term that depends on h_o includes also an exponentially decreasing factor, it is straightforward to show that this term will not affect the behavior of p and $dp/d\sigma$.

Summarizing then, since the same behavior for $\sigma \to \infty$ is obtained as for the single dimensional case, the same conclusion follows: The optimal guidance law as defined in Eq. (21) with $c = -t_f$ achieves a soft rendezvous at $r = r_f$ provided $t_f < t_f^*$.

The landing approach angle α is defined by

$$\alpha = a\ \tan(r\dot{\theta}/\dot{r}) \tag{40}$$

In Appendix B it is shown that the final landing approach angle α verifies

$$\lim_{t \to t_f} \alpha = 0 \tag{41}$$

The landing trajectory is tangent to the radius vector, or in other terms, the final landing approach is made along the local vertical.

Now, the acceleration at landing

$$a_{tf} = \lim_{t \to t_f} \frac{2v}{t - t_f} \tag{42}$$

Employing the rule of L'Hopital

$$a_{tf} = \lim_{t \to t_f} 2\dot{v} = 2\left(a_{tf} - \frac{\mu r_f}{r_f^3}\right) \tag{43}$$

Rearranging

$$a_{tf} = \frac{2\mu r_f}{r_f^3} \tag{44}$$

The required acceleration at landing is equal to twice the acceleration of gravity.

Finally a remark is to be made on the relation between the results presented here and the case of a null gravitational field, where the equations of motion as well as the adjoint equations are linear. The linear case was studied in the past and fully solved in analytical terms for a fixed final time.[10] In the linear final fixed time case, the cost J_f, as shown in Fig. 3, is a monotonically decreasing function of time of the rendezvous duration t_f, the optimum duration is infinite, and the corresponding cost null. The central gravitational field case can be reduced to the linear case by making $\mu \to 0$. The results therefore should be the same as for the linear case, except for the fact that a free final time was considered here. In Appendix C it is shown that the optimal acceleration as given in Eq. (21) is identical to the one obtained for the linear case, with a rendezvous final time $\bar{t}_f$, obtained as the limiting case of the nonlinear problem, corresponding precisely to the inflection point of J_f, as shown in Fig. 3.

VII. Numerical Results

As an application example the case of landing a spacecraft parked in an elliptic orbit about a spherical asteroid with a 20 km diameter and gravitational constant $\mu = 3900\ km^3/h^2$ is considered. The optimal guidance law has to achieve a soft landing at $r_f = 10$ km.

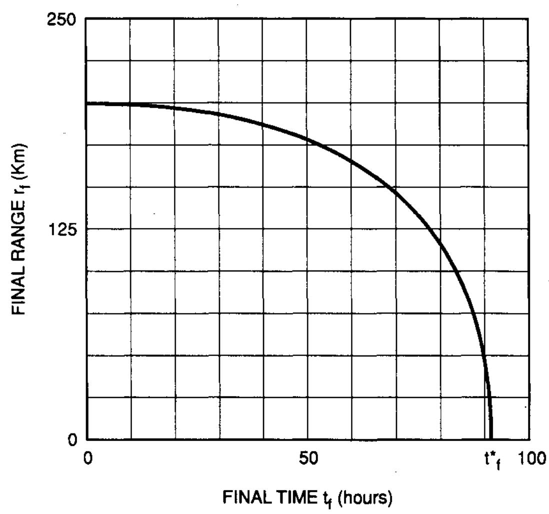

Fig. 4 Final range r_f vs final time t_f.

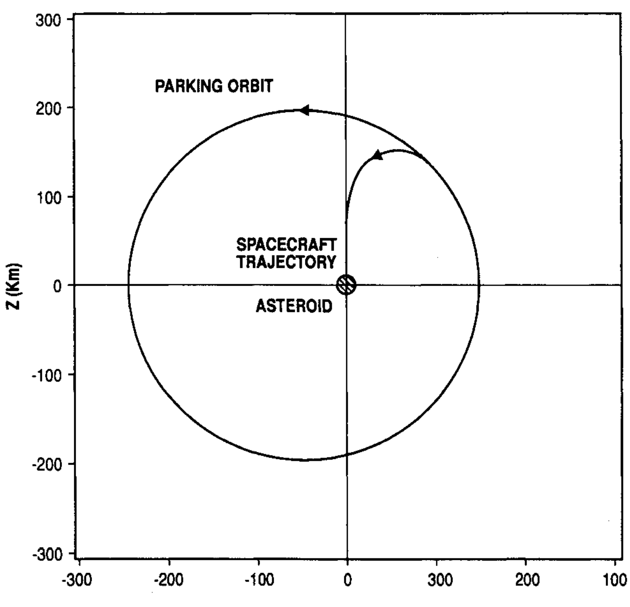

Fig. 5 Spacecraft trajectory.

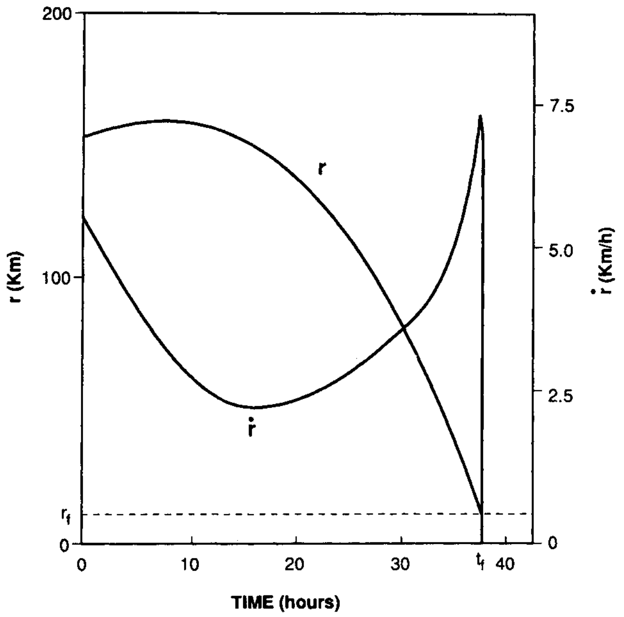

Fig. 6 Spacecraft asteroid relative range and velocity vs time.

As already stated there is no closed form solution for the equations of motion when the optimal acceleration as defined by Eq. (21) is used. However, for $c=-t_f$, and with given initial values of range and velocity r_o, v_o, the final range r_f at $t=t_f$, as obtained from the numerical solution of Eq. (29), is solely a function of t_f. As an example the case $r_o=200$ km, $\dot{r}_o=0$ km/h, $\dot{\theta}_o=0.221$ rad/h is depicted in Fig. 4. This curve has a vertical slope at $t=t_f^*$.

Given the relation between r_f and t_f, to achieve a specific final range $\bar{r}_f$ it is necessary to determine at guidance initiation the corresponding value of $\bar{\bar{t}}_f$. For given initial conditions r_o, $\dot{r}_o$, $\dot{\theta}_o$ the value of $\bar{\bar{t}}_f$ is obtained numerically as a root of $r_f(t_f)-\bar{r}_f=0$. To assure convergence of the root computation, specially taking into account the vertical slope at $t_f=t_f^*$, either t_f^*, or a smaller but close value of t_f^* is first to be determined. A close value t_{fl}^* is readily determined as the intersection of the parametric curve $p_o(t_f)$, $\mathrm{d}p/\mathrm{d}\sigma|_o(t_f)$ previously defined for the one-dimensional case ($h_o=0$), and the separatrix S_1. Because the term depending on h_o in Eq. (29) has a constant negative sign, t_{fl}^* is indeed such that $t_{fl}^*\leq t_f^*$.

An iterative Newton-Raphson algorithm is then employed, with initial values for $\bar{\bar{t}}_f=\bar{\bar{t}}_{fo}$, $\bar{\bar{t}}_{f1}<t_{fl}^*$.

The spacecraft trajectory is depicted in Fig. 5. Range and velocity relative to the asteroid are depicted in Fig. 6. As can be seen, the spacecraft effectively achieves a soft landing along the local vertical. The optimal guidance law first reduces range, even at the expense of increasing the vehicle velocity, and only toward the final approach is velocity brought to zero.

VIII. Summary and Conclusions

A particular solution was found in analytic terms for the free time optimal acceleration required for landing on an asteroid with a power limited electrically propelled spacecraft. This solution took into account the gravitational forces due to the asteroid. A qualitative analysis of the spacecraft trajectories was performed, and the optimal acceleration was shown to assure a vertical landing. A comparison was made with the well known solution of the fixed time linear case when gravitation effects are neglected. The optimal acceleration is shown to be identical to the solution obtained in the linear case for a final time corresponding to the local minimum of the cost. The solution here obtained is in feedback form providing for easy implementation. Numerical results were presented for landing on a particular asteroid.

Appendix A: Adjoint Variables Solution Verification

Differentiating p_v as defined in Eq. (20) with respect to time and introducing into Eq. (9)

$$p_r=\frac{2\mu r/r^3}{t+c}-\frac{2v}{(t+c)^2} \tag{A1}$$

Differentiating p_r with respect to time

$$\dot{p}_r=\frac{2\mu}{r^3}\left[\frac{v}{(t+c)}-\frac{3(r\cdot v)r}{(t+c)r^2}\right] \tag{A2}$$

Once p_v from Eq. (20) is introduced into Eq. (8) it can be readily seen that it is identical to Eq. (A2), effectively verifying the adjoint equations.

Appendix B: Final Landing Approach Angle

The final landing approach angle is defined by

$$\alpha_f=\lim_{t\to t_f}(r\dot{\theta}/\dot{r}) \tag{B1}$$

Since both the radial and transversal velocity components approach zero at landing, it is necessary to look into their respective behavior for $t\to t_f$ to determine α_f.

From Eq. (37) the radial velocity $\dot{r}$ is given by

$$r=t_f^{-1/3}ze^{\sigma/3} \tag{B2}$$

where

$$z=\frac{\mathrm{d}p}{\mathrm{d}\sigma}-\frac{2}{3}p \tag{B3}$$

Differentiating Eq. (B3) with respect to σ

$$\frac{\mathrm{d}z}{\mathrm{d}\sigma}=\frac{\mathrm{d}^2p}{\mathrm{d}\sigma^2}-\frac{2}{3}\frac{\mathrm{d}p}{\mathrm{d}\sigma} \tag{B4}$$

Substituting $\mathrm{d}p/\mathrm{d}\sigma$ and $\mathrm{d}^2p/\mathrm{d}\sigma^2$ from (B3) and (B4) into Eq. (34) and rearranging, a first-order differential equation for z is obtained

$$\frac{\mathrm{d}z}{\mathrm{d}\sigma}+\frac{7}{3}z=-\frac{\mu}{p^2}+\frac{h_o^2t_f^{-5/3}e^{-7\sigma/3}}{p^3} \tag{B5}$$

The solution of Eq. (B5) for $z(\sigma)$ is given by

$$z(\sigma)=\left\{z(\sigma_o)+\int_{\sigma_o}^{\sigma}\left[h_o^2t_f^{-5/3}p^{-3}(\tau)-e^{7\tau/3}\mu p^{-2}(\tau)\right]\mathrm{d}\tau\right\}e^{-7\sigma/3} \tag{B6}$$

Using now the L'Hopital rule for $\sigma\to\infty$ it is obtained

$$\lim_{\sigma\to\infty}z(\sigma)=\lim_{\sigma\to\infty}\frac{h_o^2t_f^{-5/3}p^{-3}(\sigma)-e^{7\tau/3}\mu p^{-2}(\sigma)}{(7/3)e^{7\tau/3}} \tag{B7}$$

As mentioned, p has in terms of σ an exponential order of ⅔. From Eqs. (B7) it follows that for $\sigma\to\infty$, $z(\sigma)$ behaves as $e^{-4\sigma/3}$, and from Eq. (B4), $\dot{r}$ behaves as $e^{-\sigma}$. Consequently, from Eq. (33) it is obtained that for $t\to t_f$, r behaves as (t_f-t).

The transversal velocity can be rewritten in terms of the angular momentum h as

$$r\dot{\theta}=h/r \tag{B8}$$

Taking into account Eq. (27) with $c=-t_f$, it follows that

$$\lim_{t\to t_f}r\dot{\theta}=\frac{h_o}{r_ft_f^2}\lim_{t\to t_f}(t_f-t)^2 \tag{B9}$$

Since it was already shown that $\dot{r}$ behaves as (t_f-t), Eq. (B9) implies that

$$\lim_{t\to t_f}(r\dot{\theta}/\dot{r})=0 \tag{B10}$$

Appendix C: Null Gravitational Field

In the absence of a gravitational field, the equations of motion as well as the adjoint equations are linear. The optimal acceleration for this case with the same optimality index as defined in Eq. (12), $v_f = 0$ and final time t_f fixed is given by Ref. 9

$$a_L = -\frac{6}{(t_f - t)^2}(r - r_f) - \frac{4}{(t_f - t)} v \tag{C1}$$

The corresponding cost is

$$J_f = \left(\frac{2}{t_f^3}\right)\left[t_f^2 v_o^2 + 3 t_f v_o \cdot (r_o - r_f) + 3(r_o - r_f)^2\right] \tag{C2}$$

The central gravitaional field case can be reduced to the linear case by making $\mu \to 0$.

Introducing the acceleration a^* from Eq. (21) into Eq. (2) with $\mu = 0$

$$\frac{dv}{dt} = \frac{-2v}{(t_f - t)} \tag{C3}$$

Integrating this equation from initial conditions $t = 0$, $v = v_o$

$$v = v_o \left(\frac{t_f - t}{t_f}\right)^2 \tag{C4}$$

Introducing now v from Eq. (C4) into Eq. (1) and integrating backward in time

$$r - r_f = -(1/3) v (t_f - t) \tag{C5}$$

Now, a^* from Eq. (21) can be rewritten as

$$a^* = -4 \frac{v}{t_f - t} + 2 \frac{v}{t_f - t} \tag{C6}$$

Introducing v as obtained from Eq. (C5) into the second term of the right-hand side of Eq. (C6), it can be readily shown that the optimal acceleration as given in Eq. (21) is identical to the one obtained for the linear case as defined in Eq. (C1).

For the linear case, the vehicle trajectory is a straight line along the initial velocity vector. For $t = 0$, $r = r_o$, and $v = v_o$, it follows from Eq. (C5) that

$$t_f = -3 \frac{(r_o - r_f) \cdot v_o}{v_o \cdot v_o} \tag{C7}$$

This is the rendezvous final time when use is made of the optimal acceleration as defined in Eq. (21). This particular value of t_f corresponds to the inflection point of the cost given in Eq. (C2).[11]

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